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Branching Rules for Finite-Dimensional $\mathcal{U}_q(\mathfrak{su}(3))$-Representations with Respect to a Right Coideal Subalgebra

Noud Aldenhoven[1] · Erik Koelink[1] · Pablo Román[2]

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Abstract We consider the quantum symmetric pair $(\mathcal{U}_q(\mathfrak{su}(3)), \mathcal{B})$ where $\mathcal{B}$ is a right coideal subalgebra. We prove that all finite-dimensional irreducible representations of $\mathcal{B}$ are weight representations and are characterised by their highest weight and dimension. We show that the restriction of a finite-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{su}(3))$ to $\mathcal{B}$ decomposes multiplicity free into irreducible representations of $\mathcal{B}$. Furthermore we give explicit expressions for the highest weight vectors in this decomposition in terms of dual q-Krawtchouk polynomials.

Keywords Quantum groups · Coideal subalgebras · Quantum symmetric pairs · Branching rules

1 Introduction

The theory of quantum symmetric pairs of Lie groups has been developed by Koornwinder, Dijkhuizen, Noumi and Sugitani and others [4, 5, 23–25, 28] for classical Lie groups and

Presented by Catharina Stroppel.

✉ Pablo Román
roman@famaf.unc.edu.ar

Noud Aldenhoven
n.aldenhoven@math.ru.nl

Erik Koelink
e.koelink@math.ru.nl

1 IMAPP, Radboud Universiteit, PO Box 9010, 6500 GL Nijmegen, The Netherlands

2 CIEM, FaMAF, Universidad Nacional de Córdoba, Medina Allende s/n Ciudad Universitaria, Córdoba, Argentina

by G. Letzter [16, 18–21] for all semisimple Lie algebras, see also [13]. The motivating example for the development for this theory was given by Koornwinder [15], who studied scalar-valued spherical functions on the quantum analogue of $(\mathrm{SU}(2), \mathrm{U}(1))$ considering twisted primitive elements in the quantised universal enveloping algebra of $\mathcal{U}_q(\mathfrak{sl}(2))$. Koornwinder identified all scalar-valued spherical functions with Askey-Wilson polynomials in two free parameters. Dijkhuizen and Noumi [4] extended the work of Koornwinder to quantum analogues of $(\mathrm{SU}(n+1), \mathrm{U}(n))$ considering two-sided coideals of the quantised universal enveloping algebra of $\mathcal{U}_q(\mathfrak{gl}(n+1))$. More generally, Letzter considered the quantised universal enveloping algrebra $\mathcal{U}_q(\mathfrak{g})$ with a right coideal subalgebra $\mathcal{B}$, which is the quantum analogue of $\mathcal{U}(\mathfrak{k})$ for a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In [20] all scalar-valued spherical functions for quantum symmetric pairs with reduced restricted root systems are identified with Macdonald polynomials. However, the requirement of having a reduced restricted root system excludes the quantum analogue of $(\mathrm{SU}(3), \mathrm{U}(2))$.

One recent extension of this situation [1], where higher-dimensional representations of the coideal subalgebra $\mathcal{B}$ are involved, arises with the study of matrix-valued spherical functions of the quantum analogue of $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$ where the subgroup is diagonally embedded. The quantum symmetric pair is given by the quantised universal enveloping algebra of $\mathcal{U}_q(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and a right coideal subalgebra $\mathcal{B}$ that can be identified with $\mathcal{U}_q(\mathfrak{su}(2))$. As in the Lie group setting [8, 11, 12, 27], the explicit knowledge of the branching rules plays a fundamental role in the explicit determination of the matrix-valued spherical functions. In this first case, the branching rules for the irreducible representations of $\mathcal{U}_q(\mathfrak{g})$ with respect to $\mathcal{B}$ follow using the standard Clebsch-Gordan decomposition.

One of the first technical difficulties that one runs into in order to extend the results of [1] to more general quantum symmetric pairs is the lack of the explicit branching rules for finite-dimensional $\mathcal{U}_q(\mathfrak{g})$-representations with respect to a right coideal subalgebra. In this paper we deal with this problem for the quantised universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(3))$ with a right coideal subalgebra $\mathcal{B}$ as in Kolb [13]. We study the problem of describing all irreducible representations that occur in the restriction to $\mathcal{B}$ of finite-dimensional irreducible representations of $\mathcal{U}_q(\mathfrak{su}(3))$. In general, information about branching rules for quantum symmetric pairs $(\mathcal{U}_q(\mathfrak{g}), \mathcal{B})$ as in Kolb [13] and Letzter [16, 18] is relatively scarce in particular in case the coideal subalgebra depends on an additional parameter as in this paper. However see Oblomkov and Stokman [26] for partial information on the branching rules for the quantum analogue of $(\mathfrak{gl}(2n), \mathfrak{gl}(n) \oplus \mathfrak{gl}(n))$. It would be of interest to see whether the results of Mudrov [22] can be used as well in the setting of Oblomkov and Stokman [26] to find precise information on the branching rule for this quantum symmetric pair, or more generally for quantum symmetric pairs involving the quantised universal enveloping algebra of type A.

There are various other instances where quantum symmetric pairs play an important role apart from the relation to special functions and matrix-valued orthogonal polynomials alluded to above. In the works of Ehrig and Stroppel, see e.g. [6] and Bao, see e.g. [2], it is indicated how modern representation theory for quantum groups, such as Schur-Jimbo duality, canonical bases, Kazhdan-Lusztig theory, categorification, etc., can be extended to quantum symmetric pairs. Moreover, the role of quantum symmetric pairs in integrable systems in mathematical physics is explained in [3] in conjuction with other types of boundary conditions. Then they are known as boundary quantum groups and they are related to explicit solutions of the reflection equation.

This paper is organised as follows. In Section 2 we review the construction of the quantised universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(3))$ and its finite dimensional irreducible representations. Then we collect a series of commutation identities for the generators of $\mathcal{U}_q(\mathfrak{su}(3))$ and we introduce an orthogonal basis for finite-dimensional $\mathcal{U}_q(\mathfrak{su}(3))$-representations which is an analogue of Mudrov [22]. We also describe the action of the generators of $\mathcal{U}_q(\mathfrak{su}(3))$ on this basis. In Section 3 we fix a right coideal subalgebra $\mathcal{B}$ of the quantised universal enveloping algebra which depends on two complex parameters c_1, c_2. We describe the generators of the Cartan subalgebra of $\mathcal{B}$ and we use them to classify all finite-dimensional irreducible representations of $\mathcal{B}$ under a mild genericity condition on the parameters. More precisely we prove that every finite-dimensional irreducible representation of $\mathcal{B}$ is completely characterised by its highest weight and its dimension. In Section 4 we prove the main theorem of the paper. We show that any irreducible finite-dimensional representation of $\mathcal{U}_q(\mathfrak{su}(3))$ decomposes multiplicity free into irreducible representations of $\mathcal{B}$ and we characterise the representations that occur in the decomposition by their highest weight and dimension. The highest weight vectors of the coideal subalgebra $\mathcal{B}$-representations are obtained by diagonalising an element of the Cartan subalgebra of $\mathcal{B}$ restricted to a certain subspace where it acts tridiagonally. The eigenvectors can be then identified explicitly in terms of dual q-Krawtchouk polynomials.

2 The Quantised Universal Enveloping Algebra $\mathcal{U}_q(\mathfrak{su}(3))$

Let $\mathfrak{g} = \mathfrak{sl}(3) = \{X \in \mathfrak{gl}(3,\mathbb{C}) : \operatorname{tr}(X) = 0\}$. We fix the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices. Let $A = (a_{i,j})_{i,j}$ be the Cartan matrix for $\mathfrak{g}$, i.e. $a_{i,i} = 2$, $i = 1, 2$, and $a_{i,j} = -1$ for $i \neq j$. Let $R \subset \mathfrak{h}$ denote the root system of $\mathfrak{g}$. We denote by R^+ the subset of positive roots, so that we have the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. We denote by $(\cdot,\cdot)$ the canonical inner product on $\mathfrak{h}$ and by $\Pi = \{\alpha_1, \alpha_2\}$ the simple roots so that $(\alpha_i, \alpha_j) = a_{i,j}$. The fundamental weights are given by $\varpi_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$ and $\varpi_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$.

The quantised universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(3))$ is the unital associative algebra over $\mathbb{C}$ generated by E_i, F_i and $K_i^{\pm 1}$, where $i = 1, 2$, subject to the relations

$$
\begin{aligned}
K_i^{\pm1}K_j^{\pm1} &= K_j^{\pm1}K_i^{\pm1}, \quad K_i^{\pm1}K_j^{\mp1} = K_j^{\mp1}K_i^{\pm1}, \quad K_iK_i^{-1} = 1 = K_i^{-1}K_i,\\
K_iE_j &= q^{(\alpha_i,\alpha_j)}E_jK_i, \quad K_iF_j = q^{-(\alpha_i,\alpha_j)}F_jK_i, \quad \left[E_i, F_j\right] = \tfrac{K_i - K_i^{-1}}{q-q^{-1}}\delta_{i,j},
\end{aligned}
\tag{2.1}
$$

for $i, j = 1, 2$ and, for $i \neq j$, the quantum Serre's relations

$$
E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0 = F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2. \tag{2.2}
$$

We assume that $q \in (0, 1)$. The quantised universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(3))$ has a Hopf algebra structure with comultiplication Δ, counit ϵ and antipode S defined by

$$
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i^{\pm1}) = K_i^{\pm1} \otimes K_i^{\pm1},
$$

$$
\epsilon(E_i) = 0, \quad \epsilon(F_i) = 0, \quad \epsilon(K_i^{\pm1}) = 1,
$$

$$
S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i, \quad S(K_i^{\pm1}) = K_i^{\mp1},
$$

with $i = 1, 2$. With the $*$-structure given by

$$
E_i^* = K_iF_i, \quad F_i^* = E_iK_i^{-1}, \quad (K_i^{\pm1})^* = K_i^{\pm1}, \quad i = 1, 2, \tag{2.3}
$$

$\mathcal{U}_q(\mathfrak{sl}(3))$ becomes a Hopf $*$-algebra which we denote by $\mathcal{U}_q(\mathfrak{su}(3))$. Following Mudrov [22] we define for $a \in \mathbb{R}$

$$\begin{aligned} F_3 &= [F_1, F_2]_q = F_1F_2 - qF_2F_1, \quad E_3 = [E_2, E_1]_q = E_2E_1 - qE_1E_2, \\ \hat{F}_3[a] &= F_1F_2\left(\frac{q^{a+1}K_2 - q^{-a-1}K_2^{-1}}{q-q^{-1}}\right) - F_2F_1\left(\frac{q^aK_2 - q^{-a}K_2^{-1}}{q-q^{-1}}\right), \\ \hat{E}_3[a] &= \left(\frac{q^{a+1}K_2 - q^{-a-1}K_2^{-1}}{q-q^{-1}}\right)E_2E_1 - \left(\frac{q^aK_2 - q^{-a}K_2^{-1}}{q-q^{-1}}\right)E_1E_2, \end{aligned}$$

and $\hat{F}_3 = \hat{F}_3[0]$, $\hat{E}_3 = \hat{E}_3[0]$.

Lemma 2.1 *The following relations hold in $\mathcal{U}_q(\mathfrak{su}(3))$:*

(i) $F_1\hat{F}_3[a] = \hat{F}_3[a]F_1$,
(ii) $E_2\hat{F}_3[a] = \hat{F}_3[a-2]E_2 - E_2\hat{F}_3[a] = \hat{F}_3[a-2]E_2 - \frac{(q^a - q^{-a})}{(q-q^{-1})}F_1$,
(iii) $K_i\hat{F}_3[a] = q^{-1}\hat{F}_3[a]K_i$, $K_i\hat{E}_3[a] = q\hat{E}_3[a]K_i$, $i = 1, 2$.

Proof Straightforward verifications using (2.1) and (2.3). □

Lemma 2.2 *For $i = 1, 2$:*

(i)
$$E_iF_i^k = F_i^kE_i + \frac{q^k - q^{-k}}{q-q^{-1}}F_i^{k-1}\frac{q^{1-k}K_i - q^{k-1}K_i^{-1}}{q-q^{-1}}$$

(ii)
$$\begin{aligned} E_i^kF_i^k &= \frac{q^k(q^2;q^2)_k}{(1-q^2)^{2k}}(q^{2-2k}K_i^2;q^2)_kK_i^{-k} + X\,E_i \\ &= \frac{(q^2;q^2)_k}{(1-q^2)^{2k}}(-1)^kq^{-k(k-2)}(K_i^{-2};q^2)_kK_i^k + X\,E_i, \end{aligned}$$

for some $X \in \mathcal{U}_q(\mathfrak{su}(3))$.

Proof Straightforward verifications using (2.1) and (2.3) and induction. □

2.1 The Finite-Dimensional Representations of $\mathcal{U}_q(\mathfrak{su}(3))$

Finite-dimensional representations of $\mathcal{U}_q(\mathfrak{su}(3))$ are weight representations and are uniquely determined, up to equivalence, by their highest weights. Let (π_λ, V_λ) be the irreducible finite-dimensional representation with highest weight $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2$, $\lambda_1, \lambda_2 \in \mathbb{N}$, and v_λ a highest weight vector, i.e. a non-zero vector $v_\lambda \in V_\lambda$ such that

$$E_i\,v_\lambda = 0, \quad K_i\,v_\lambda = q^{(\lambda,\alpha_i)}v_\lambda = q^{\lambda_i}v_\lambda, \qquad i = 1, 2. \tag{2.4}$$

Then the dimension of V_λ is the same as the dimension of the corresponding irreducible representation π_λ of $\mathfrak{su}(3)$, namely

$$\dim(V_\lambda) = \frac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2).$$

Furthermore for a weight $\nu = \nu_1\varpi_1 + \nu_2\varpi_2$, the dimension of the weight spaces

$$V_\lambda(\nu) = \{v \in V_\lambda :\; K_i\,v = q^{(\nu,\alpha_i)}v,\; i = 1, 2\},$$

and the dimension of the weight spaces corresponding to the weight ν in the representation of $\mathfrak{su}(3)$ coincide, see [9, Ch. 7]. In particular, $\dim(V_\lambda(\lambda)) = 1$. The vector space V_λ is spanned by the vectors $F_{i_1} F_{i_2} \cdots F_{i_m} v_\lambda$, $i_j \in \{1,2\}$ and is equipped with an inner product $\langle\cdot,\cdot\rangle$ determined by

$$\langle v_\lambda, v_\lambda\rangle = 1, \qquad \langle X\,v, w\rangle = \langle v, X^*\,w\rangle, \qquad \forall\, X \in \mathcal{U}_q(\mathfrak{su}(3)), \quad \forall\, v, w \in V_\lambda.$$

Mudrov [22] describes the Shapovalov basis for the Verma modules of $\mathcal{U}_q(\mathfrak{su}(3))$, and we have adapted his proof and construction to an orthonormal basis for the finite-dimensional unitary representations of $\mathcal{U}_q(\mathfrak{su}(3))$. For completeness, we have sketched the proof in Appendix A. It is essentially due to Mudrov [22, §8].

Theorem 2.3 *The set of vectors*

$$\mathscr{B} = \{F_2^k \hat{F}_3^l F_1^m\, v_\lambda \mid 0 \le m \le \lambda_1,\ 0 \le l \le \lambda_2,\ 0 \le k \le \lambda_2 + m - l\}$$

forms an orthogonal basis for V_λ. Explicitly,

$$\langle F_2^k \hat{F}_3^l F_1^m\, v_\lambda, F_2^{k'} \hat{F}_3^{l'} F_1^{m'}\, v_\lambda\rangle = \delta_{k,k'}\delta_{l,l'}\delta_{m,m'} H_{k,l,m},$$

where

$$\begin{aligned} H_{k,l,m} = {} & (q^2, q^{-2(\lambda_2-l+m)}; q^2)_k (q^2, q^{-2\lambda_1}; q^2)_m (q^2, q^{-2\lambda_2}, q^{-2(\lambda_2+1+m)}, \\ & q^{-2(\lambda_1+\lambda_2+1)}; q^2)_l \times (1-q^2)^{-2(k+2l+m)} (-1)^{k+l+m} q^{3(k+3l+m)} q^{-l(l-2m)} q^{-2l\lambda_2}. \end{aligned}$$

In Theorem 2.3 we use the standard notation in [7] for q-shifted factorials

$$\begin{aligned} (q^a; q)_n &= (1-q^a)(1-q^{a+1})\cdots(1-q^{a+n-1}), \\ (q^{a_1}, q^{a_2}, \cdots, q^{a_j}; q)_n &= (q^{a_1}; q)_n (q^{a_2}; q)_n \cdots (q^{a_j}; q)_n, \end{aligned}$$

switching from base q to base q^2. Note that $H_{k,l,m}$ is indeed positive. In the following proposition we calculate the action of the generators of $\mathcal{U}_q(\mathfrak{su}(3))$ in the basis $\mathscr{B}$ of Theorem 2.3.

Proposition 2.4 *In the basis $\mathscr{B}$ of V_λ as in Theorem 2.3 we have*

(i) $K_1 F_2^k \hat{F}_3^l F_1^m\, v_\lambda = q^{\lambda_1+k-l-2m}\, F_2^k \hat{F}_3^l F_1^m\, v_\lambda$,
(ii) $K_2 F_2^k \hat{F}_3^l F_1^m\, v_\lambda = q^{\lambda_2-2k-l+m}\, F_2^k \hat{F}_3^l F_1^m\, v_\lambda$,
(iii) $F_1 F_2^k \hat{F}_3^l F_1^m\, v_\lambda = a_k(l,m)\, F_2^k \hat{F}_3^l F_1^{m+1}\, v_\lambda + b_k(l,m)\, F_2^{k-1} \hat{F}_3^{l+1} F_1^m\, v_\lambda$,
(iv) $E_1 F_2^k \hat{F}_3^l F_1^m\, v_\lambda = \alpha_k(l,m)\, F_2^k \hat{F}_3^l F_1^{m-1}\, v_\lambda + \beta_k(l,m)\, F_2^{k+1} \hat{F}_3^{l-1} F_1^m\, v_\lambda$,
(v) $F_2 F_2^k \hat{F}_3^l F_1^m\, v_\lambda = F_2^{k+1} \hat{F}_3^l F_1^m\, v_\lambda$,
(vi) $E_2 F_2^k \hat{F}_3^l F_1^m\, v_\lambda = \eta_k(l,m)\, F_2^{k-1} \hat{F}_3^l F_1^m\, v_\lambda$,

with coefficients

$$a_k(l,m) = \frac{(q^{\lambda_2+m+1-k-l} - q^{-\lambda_2-m-1+k+l})}{(q^{\lambda_2+m+1-l} - q^{-\lambda_2-m-1+l})}, \quad b_k(l,m) = \frac{(q^k - q^{-k})}{(q^{\lambda_2+m+1-l} - q^{-\lambda_2-m-1+l})},$$

$$\eta_k(l,m) = \frac{q^k - q^{-k}}{q - q^{-1}} \frac{q^{1-k+\lambda_2-l+m} - q^{k-1-\lambda_2+l-m}}{q - q^{-1}},$$

$$\alpha_k(l,m) = \frac{(q^m - q^{-m})(q^{\lambda_1-m+1} - q^{-\lambda_1+m-1})(q^{\lambda_2+m+1} - q^{-\lambda_2-m-1})}{(q-q^{-1})^2 (q^{\lambda_2+m-l+1} - q^{-\lambda_2-m+l-1})},$$

$$\beta_k(l,m) = \frac{(q^l - q^{-l})(q^{\lambda_2-l+1} - q^{-\lambda_2+l-1})(q^{\lambda_1+\lambda_2-l+2} - q^{-\lambda_1-\lambda_2+l-2})}{(q-q^{-1})^2 (q^{\lambda_2+m-l+1} - q^{-\lambda_2-m+l-1})}.$$

Remark 2.5 Note that the denominators in $a_k(l,m)$, $b_k(l,m)$, $\eta_k(l,m)$, $\alpha_k(l,m)$ and $\beta_k(l,m)$ are non-zero by the ranges of k,l,m as in Theorem 2.3.

Proof The action of K_i, $i=1,2$, follows from (2.4), (2.1) and Lemma 2.11(iii). The action of F_2 is trivial. The action of E_2 follows from Lemma 2.2(i), Lemma 2.1(ii) and (2.4) and the established actions of K_2. This completes the proof of (i), (ii), (v) and (vi).

In order to establish the action of F_1, we first show that there exist constants a_k and b_k so that

$$F_1 F_2^k \hat{F}_3^l F_1^m\, v_\lambda = a_k F_2^k \hat{F}_3^l F_1^{m+1}\, v_\lambda + b_k F_2^{k-1} \hat{F}_3^{l+1} F_1^m\, v_\lambda$$

by induction with respect to k. The case $k=0$ with $a_0=1$, $b_0=0$ is immediate from Lemma 2.1(i). In case $k=1$, we write

$$\begin{aligned}
F_1 F_2 \hat{F}_3^l F_1^m\, v_\lambda &= F_1 F_2 \frac{qK_2 - q^{-1}K_2^{-1}}{q-q^{-1}} \frac{q-q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}} \hat{F}_3^l F_1^m\, v_\lambda \\
&= \frac{q-q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}} \left(\hat{F}_3 + F_2 F_1 \frac{K_2 - K_2^{-1}}{q-q^{-1}} \right) \hat{F}_3^l F_1^m\, v_\lambda \\
&= \frac{q^{\lambda_2-l+m} - q^{-\lambda_2+l-m}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}} F_2 \hat{F}_3^l F_1^{m+1}\, v_\lambda \\
&\qquad + \frac{q-q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}} \hat{F}_3^{l+1} F_1^m\, v_\lambda
\end{aligned}$$

again using Lemma 2.1(i). So the case $k=1$ is proved with

$$a_1 = \frac{q^{\lambda_2-l+m} - q^{-\lambda_2+l-m}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}}, \qquad b_1 = \frac{q-q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2+l-m}}.$$

For the induction we assume $k \geq 2$, so that

$$F_1 F_2^k \hat{F}_3^l F_1^m\, v_\lambda = F_1 F_2^2 F_2^{k-2} \hat{F}_3^l F_1^m\, v_\lambda = \big(-F_2^2 F_1 + (q+q^{-1}) F_2 F_1 F_2\big) F_2^{k-2} \hat{F}_3^l F_1^m\, v_\lambda$$

by the q-Serre relation (2.2). Using the induction hypothesis, we find

$$\begin{aligned}
F_1 F_2^k \hat{F}_3^l F_1^m\, v_\lambda &= -F_2^2 \big(a_{k-2} F_2^{k-2} \hat{F}_3^l F_1^{m+1}\, v_\lambda + b_{k-2} F_2^{k-3} \hat{F}_3^{l+1} F_1^m\, v_\lambda\big) \\
&\quad + (q+q^{-1}) F_2 \big(a_{k-1} F_2^{k-1} \hat{F}_3^l F_1^{m+1}\, v_\lambda + b_{k-1} F_2^{k-2} \hat{F}_3^{l+1} F_1^m\, v_\lambda\big) \\
&= (-a_{k-2} + (q+q^{-1}) a_{k-1}) F_2^k \hat{F}_3^l F_1^{m+1}\, v_\lambda + (-b_{k-2} + (q+q^{-1}) b_{k-1}) F_2^{k-1} \hat{F}_3^{l+1} F_1^m\, v_\lambda
\end{aligned}$$

which proves the induction step as well as the recurrence

$$a_k + a_{k-2} = (q+q^{-1}) a_{k-1}, \qquad b_k + b_{k-2} = (q+q^{-1}) b_{k-1}, \qquad k \geq 2.$$

This recursion is solved by the Chebyshev polynomials (of the second kind) at $\frac{1}{2}(q+q^{-1})$ as well as by the associated Chebyshev polynomials. This gives the solution for the recurrences and proves (iii).

The action of E_1 follows from that of F_1, considering the adjoint. Note that

$$\begin{aligned}
\langle E_1 F_2^k \hat{F}_3^l F_1^m\, v_\lambda, F_2^{k'} \hat{F}_3^{l'} F_1^{m'}\, v_\lambda \rangle &= \langle F_2^k \hat{F}_3^l F_1^m\, v_\lambda, E_1^* F_2^{k'} \hat{F}_3^{l'} F_1^{m'}\, v_\lambda \rangle \\
&= \langle F_2^k \hat{F}_3^l F_1^m\, v_\lambda, K_1 F_1 F_2^{k'} \hat{F}_3^{l'} F_1^{m'}\, v_\lambda \rangle,
\end{aligned}$$

equals zero if $(k', l', m') \neq (k, l, m+1), (k+1, l-1, m)$. Moreover we have

$$\begin{aligned}
\alpha_k(l,m) H_{k,l,m-1} &= \langle E_1 F_2^k \hat{F}_3^l F_1^m \, v_\lambda, \, F_2^k \hat{F}_3^l F_1^{m-1} \, v_\lambda \rangle \\
&= \langle F_2^k \hat{F}_3^l F_1^m \, v_\lambda, \, K_1 F_1 F_2^k \hat{F}_3^l F_1^{m-1} \, v_\lambda \rangle \\
&= q^{k-l-2m+\lambda_1} \, a_k(l, m-1) \, H_{k,l,m},
\end{aligned}$$

and

$$\begin{aligned}
\beta_k(l,m) H_{k+1,l-1,m} &= \langle E_1 F_2^k \hat{F}_3^l F_1^m \, v_\lambda, \, F_2^{k+1} \hat{F}_3^{l-1} F_1^m \, v_\lambda \rangle \\
&= \langle F_2^k \hat{F}_3^l F_1^m \, v_\lambda, \, K_1 F_1 F_2^{k-1} \hat{F}_3^{l-1} F_1^m \, v_\lambda \rangle \\
&= q^{k-l-2m+\lambda_1+2} \, b_{k+1}(l-1, m) \, H_{k,l,m}.
\end{aligned}$$

Now the expressions of $\alpha_k(l,m)$ and $\beta_k(l,m)$ follow from the explicit expression of $H_{k,l,m}$ Theorem 2.3 by a straightforward computation. □

3 The Coideal Subalgebra

In this section we follow Kolb [13] and introduce a right coideal subalgebra $\mathcal{B}$ of $\mathcal{U}_q(\mathfrak{su}(3))$ which is the quantum analogue of $\mathcal{U}(\mathfrak{k})$ with $\mathfrak{k} = \mathfrak{u}(2)$ embedded in $\mathfrak{g} = \mathfrak{su}(3)$. Let $c_1, c_2 \in \mathbb{C}^\times$ and write $c = (c_1, c_2)$. Following [13, Example 9.4], $\mathcal{B}_c = \mathcal{B}$ is the right coideal subalgebra of $\mathcal{U}_q(\mathfrak{su}(3))$, i.e. $\Delta(\mathcal{B}) \subset \mathcal{B} \otimes \mathcal{U}_q(\mathfrak{su}(3))$, generated by

$$K^{\pm 1} = \left(K_1 K_2^{-1}\right)^{\pm 1}, \quad B_1^c = B_1 = F_1 - c_1 E_2 K_1^{-1}, \quad B_2^c = B_2 = F_2 - c_2 E_1 K_2^{-1}. \tag{3.1}$$

Throughout Sections 3 and 4 we omit the subscript and superscript c in $\mathcal{B}_c$ and B_i^c since the coideal subalgebra $\mathcal{B}$ will be fixed.

If we assume $c_1 \overline{c_2} = q^3 = \overline{c_1} c_2$ then it follows that $B_1^* = -\overline{c_1} K^{-1} B_2$, $B_2^* = -\overline{c_2} K B_1$ and $K^* = K$, so that $\mathcal{B}^* = \mathcal{B}$. By a straightforward computation we have

$$\begin{aligned}
\Delta(B_1) &= B_1 \otimes K_1^{-1} + 1 \otimes F_1 - c_1 K^{-1} \otimes E_2 K_1^{-1}, \\
\Delta(B_2) &= B_2 \otimes K_2^{-1} + 1 \otimes F_2 - c_2 K \otimes E_1 K_2^{-1}.
\end{aligned}$$

The Serre relations for $\mathcal{B}$ follow from from [13, Lemma 7.2, Theorem 7.4] taking $\mathcal{Z}_1 = -K^{-1}$ and $\mathcal{Z}_2 = -K$

$$\begin{aligned}
B_1^2 B_2 - [2]_q B_1 B_2 B_1 + B_2 B_1^2 &= [2]_q (q c_2 K + q^{-2} c_1 K^{-1}) B_1, \\
B_2^2 B_1 - [2]_q B_2 B_1 B_2 + B_1 B_2^2 &= [2]_q (q c_1 K^{-1} + q^{-2} c_2 K) B_2.
\end{aligned} \tag{3.2}$$

Here we use the notation $[j]_q = \frac{q^j - q^{-j}}{q - q^{-1}}$. Alternatively (3.2) can be verified directly from the definitions of B_1, B_2 and K.

Introduce $C_1, C_2 \in \mathcal{B}$ by

$$\begin{aligned}
C_1 &= B_1 B_2 - q B_2 B_1 - \tfrac{1}{q - q^{-1}} c_2 K + \tfrac{q + q^{-1}}{q - q^{-1}} c_1 K^{-1}, \\
C_2 &= B_2 B_1 - q B_1 B_2 - \tfrac{1}{q - q^{-1}} c_1 K^{-1} + \tfrac{q + q^{-1}}{q - q^{-1}} c_2 K,
\end{aligned} \tag{3.3}$$

so that for $c_1, c_2 \in \mathbb{R}^\times$, C_1 and C_2 are self-adjoint. Then the relations $[K, C_i] = 0$ for $i = 1, 2$, $[C_1, C_2] = 0$ and

$$\begin{aligned}
K B_1 &= q^{-3} B_1 K, \quad C_1 B_1 = q B_1 C_1, \quad C_2 B_1 = q^{-1} B_1 C_2, \\
K B_2 &= q^3 B_2 K, \quad C_1 B_2 = q^{-1} B_2 C_1, \quad C_2 B_2 = q B_2 C_2,
\end{aligned} \tag{3.4}$$

hold. We view the commutative algebra generated by $K^{\pm 1}$, C_1, C_2 as a Cartan subalgebra. Note that by [14, Theorem 8.5] the center of $\mathcal{B}$ is of rank 2. Hence the center of $\mathcal{B}$ is generated by $K^{\frac{1}{3}}C_1$ and $K^{-\frac{1}{3}}C_2$, extending $\mathcal{B}$ by cube roots of K. Then the central elements are self-adjoint for $c_1, c_2 \in \mathbb{R}^\times$.

3.1 Representation Theory of $\mathcal{B}$

Let (τ, W) be a finite-dimensional representation of $\mathcal{B}$. Since W is a finite-dimensional complex vector space, there exists a non-zero vector $w \in W$ such that $\tau(K)w = \nu w$ for some $\nu \in \mathbb{C}$. Then it follows from (3.4) that

$$\tau(K)\tau(B_1)^i\, w = q^{-3i}\,\tau(B_1)^i\tau(K)w = q^{-3i}\nu\tau(B_1)^i\, w, \qquad i \in \mathbb{N},$$

so that the vectors $(\tau(B_1)^i\, w)_i$ are eigenvectors of $\tau(K)$ with different eigenvalues. Since W is finite-dimensional, there exists $j \in \mathbb{N}$ such that $\tau(B_1^{j+1})w = 0$ and $\tau(B_1^j)w \neq 0$. Therefore $w_0 = \tau(B_1^j)w$ is a highest weight vector, i.e.

$$\tau(B_1)\, w_0 = 0, \qquad \tau(K)\, w_0 = \kappa\, w_0,$$

where κ is the weight of w_0. Note that $\kappa \in \mathbb{C}^\times$ since it is the eigenvalue of an invertible operator.

Proposition 3.1 *Let τ be a finite-dimensional irreducible representation of $\mathcal{B}$ on the vector space W. Then τ is determined by the dimension of W and the action of K on a highest weight vector.*

Proof Let $\kappa \in \mathbb{C}^\times$ be a highest weight of τ and let w_0 be a highest weight vector, i.e. $\tau(K)\, w_0 = \kappa\, w_0$ and $\tau(B_1)w_0 = 0$. Since $\tau(K), \tau(C_1)$ and $\tau(C_2)$ form a commuting family of operators, preserving the kernel of $\tau(B_1)$ by (3.4), we can assume that $\tau(C_1)\, w_0 = \eta_1\, w_0$ and $\tau(C_2)\, w_0 = \eta_2\, w_0$. For every $i \in \mathbb{N}$, we define the vector $w_i = \tau(B_2)^i w_0 \in W$. Since W is finite-dimensional, there exists $n \in \mathbb{N}$ such that $w_i \neq 0$ for $0 \leq i \leq n$ and $w_{n+1} = 0$. It follows from (3.4) that $\tau(K)w_i = q^{3i}\,\kappa\, w_i$, so that $(w_i)_{i=0}^n$ is a set of linearly independent vectors since they are eigenvectors of $\tau(K)$ for different eigenvalues. Moreover (3.4) implies

$$\tau(C_1)\, w_i = \tau(C_1)\tau(B_2)^i\, w_0 = q^{-i}\,\tau(B_2)^i\tau(C_1)\, w_0 = \eta_1 q^{-i}\, w_i,$$

and similarly $\tau(C_2)\, w_i = \eta_2 q^i\, w_i$. We will show that it is indeed a basis of W.

We prove by induction in i that there exist $b_i \in \mathbb{C}$ such that $\tau(B_1)w_i = b_i w_{i-1}$ for $i = 0, \ldots, n$. The statement holds for $i = 0$ taking $b_0 = 0$ since w_0 is a highest weight vector. Let $i > 0$ and assume that $\tau(B_1)w_j = b_j w_{j-1}$ for for all $j < i$. Using (3.3) we find the recurrence relation

$$\begin{aligned}\tau(B_1)w_i &= \tau(B_1)\tau(B_2)^i w_0 = \tau(B_1B_2)w_{i-1}\\ &= \tau\left(C_1 + qB_2B_1 + \tfrac{c_2}{(q-q^{-1})}K - \tfrac{(q+q^{-1})}{(q-q^{-1})}c_1K^{-1}\right) w_{i-1}\\ &= q\tau(B_2B_1)w_{i-1} + \tau\left(C_1 + \tfrac{c_2}{(q-q^{-1})}K - \tfrac{(q+q^{-1})}{(q-q^{-1})}c_1K^{-1}\right) w_{i-1}.\end{aligned}$$

By the inductive hypothesis, $\tau(B_2B_1)w_{i-1} = b_{i-1}\tau(B_2)w_{i-2} = b_{i-1}w_{i-1}$, so that

$$\tau(B_1)w_i = \left(qb_{i-1} + q^{1-i}\eta_1 + \tfrac{q^{3i-3}\kappa\, c_2}{(q-q^{-1})} - \tfrac{(q+q^{-1})}{(q-q^{-1})}q^{3-3i}\kappa^{-1}\, c_1\right) w_{i-1}. \tag{3.5}$$

Hence $\tau(B_1)w_i = b_i w_{i-1}$. Since τ is an irreducible representation we have that $W = \tau(\mathcal{B})\, w_0 = \langle\{w_0, w_1, \ldots, w_n\}\rangle$, and therefore $(w_i)_{i=0}^n$ is a basis of W. This completes the proof of the proposition. □

Remark 3.2 Since we assume (τ, W) irreducible, the coefficients b_i in the proof of Proposition 3.1 are non-zero for $i = 1, \dots, n$. This follows from the fact that if $b_{i_0} = 0$ for some $1 \le i_0 \le n$, then $\langle \{w_{i_0}, w_{i_0+1}, \dots, w_n\} \rangle$ is an invariant subspace and this contradicts the irreducibility of τ.

Corollary 3.3 *Let (τ, W) be a finite-dimensional irreducible representation of $\mathcal{B}$ of dimension $n+1$ and highest weight κ. let w_0 be a highest weight vector and let $w_i = (B_2)^i w_0$ for $i = 1, \dots, n$. Then $(w_i)_{i=0}^n$ is a basis of W. The action of the generators of $\mathcal{B}$ on this basis is given by*

$$\tau(K)\, w_j = q^{3j}\,\kappa\, w_j, \qquad \tau(B_2)\, w_j = w_{j+1}, \qquad \tau(B_1)\, w_j = b_j w_{j-1}$$

where

$$b_0 = 0, \qquad b_j = c_1\,\kappa^{-1}\, q^{-2n-1}\,[j]_q\, \frac{(1-q^{2n-2j+2})(1+c_2 c_1^{-1}\kappa^2 q^{2j+2n-1})}{(q-q^{-1})}.$$

Moreover, $\tau(C_1)\, w_j = q^{-j}\,\eta_1\, w_j$ and $\tau(C_2)\, w_j = q^j\,\eta_2\, w_j$, where

$$\eta_1 = \frac{c_1\,\kappa^{-1}\, q(1+q^{-2n-2}) - c_2\,\kappa\, q^{2n}}{q-q^{-1}}, \qquad \eta_2 = \frac{c_2\,\kappa\, q^{-1}(1+q^{2n+2}) - c_1\,\kappa^{-1}\, q^{-2n}}{q-q^{-1}}.$$

Proof The fact that $(w_i)_{i=0}^n$ is a basis of W and the action of $\tau(K)$ on w_j follow directly from the proof of Proposition 3.1. It is clear that $b_0 = 0$. We now show that

$$b_j = [j]_q \left(\eta_1 + \frac{c_2\,\kappa\, q^{2j-2} - c_1\,\kappa^{-1}\, q^{1-2j}(1+q^{2j})}{(q-q^{-1})} \right), \tag{3.6}$$

for all $j = 1, \dots, n$. We proceed by induction on i. If $i = 1$, then the statement follows directly from (3.5). Now we assume that (3.6) is true for some j, $1 < j \le n$. Then it follows from (3.5) and the inductive hypothesis that

$$\begin{aligned} b_j \;&= q\,[j-1]_q \left(\eta_1 + \frac{c_2\,\kappa\, q^{2j-4} - c_1\,\kappa^{-1}\, q^{3-2j}(1+q^{2j-2})}{(q-q^{-1})} \right) \\ &\quad + q^{1-j}\eta_1 + \frac{q^{3j-3}\kappa\, c_2}{(q-q^{-1})} - \frac{(q+q^{-1})}{(q-q^{-1})} q^{3-3j}\kappa^{-1}\, c_1. \end{aligned}$$

Now (3.6) follows by a straightforward computation.

It follows from the proof of Proposition 3.1 that $\tau(C_1)\, w_j = q^{-j}\,\eta_1\, w_j$ where η_1 is the eigenvalue for the highest weight vector w_0. From the construction of the vectors w_i in Proposition (3.1), it follows that $\tau(B_2)\, w_n = 0$. Hence (3.3) and (3.6) yield

$$\begin{aligned} q^{-n}\eta_1\, w_n \;&= \tau(C_1)\, w_n = q\tau(B_2 B_1)\, w_n - \frac{1}{q-q^{-1}} c_2\, \tau(K) w_n + \frac{q+q^{-1}}{q-q^{-1}}\, c_1\, \tau(K^{-1}) w_n \\ &= -\frac{q^{n+1}-q^{-n+1}}{q-q^{-1}}\eta_1 - \frac{q^{n+1}-q^{-n+1}}{q-q^{-1}} \left(\frac{c_2\,\kappa\, q^{2n-2} - c_2\,\kappa^{-1}\, q^{1-2n}(1+q^{2n})}{q-q^{-1}} \right) \\ &\quad - \frac{c_2\,\kappa\, q^{3n}}{q-q^{-1}} + \frac{(q+q^{-1})\, c_1\,\kappa^{-1}\, q^{-3n}}{q-q^{-1}}. \end{aligned}$$

Now the expression of η_1 follows by a straightforward computation. The expression of η_2 can be obtained similarly from the action of C_2 on w_n. $\square$

Remark 3.4 If τ is an irreducible representation with highest weight κ and dimension $n+1$, it follows from Remark 3.2 and the explicit expression of the coefficient b_i in Corollary 3.3 that $c_2c_1^{-1}\kappa^2 \neq -q^{-2j-2n+1}$ for all $j = 1, \dots, n$.

Remark 3.5 It follows from Proposition 3.1 and Corollary 3.3 that a finite-dimensional irreducible representation (τ, W) of $\mathcal{B}$ is completely determined by the highest weight κ and the eigenvalue of η_1 of the highest weight vector as eigenvector of $\tau(C_1)$.

Corollary 3.6 *Every irreducible finite-dimensional representation of $\mathcal{B}$ is determined by a pair (κ, n) where κ is the highest weight and the dimension is $n+1$. Conversely, to each pair (κ, n) with $\kappa \in \mathbb{C}^\times$, $n \in \mathbb{N}$ and $\kappa^2 \notin -c_1c_2^{-1}q^{1-\mathbb{N}}$, there corresponds an irreducible representation $(\tau_{(\kappa,n)}, W_{(\kappa,n)})$ with highest weight κ and dimension $n+1$.*

Proof It follows directly from Proposition 3.1, Corollary 3.3 and Remark 3.4. □

Proposition 3.7 *Assume that $\kappa \in \mathbb{R}^\times$ and $c_1\overline{c}_2 = q^3$. Let (τ, W) be an irreducible finite-dimensional representation of $\mathcal{B}$. Then τ is unitarizable.*

Proof Since $c_1\overline{c_2} = q^3$, we have that $\mathcal{B}^* = \mathcal{B}$. More precisely $B_1^* = -\overline{c_1}K^{-1}B_2$, $B_2^* = -\overline{c_2}KB_1$ and $K^* = K$. Let $(w_i)_{i=0}^n$ be the basis of W given in Corollary 3.3 and let $\langle\cdot,\cdot\rangle$ be the hermitian bilinear form defined on the basis elements by $\langle w_0, w_0\rangle = 1$,

$$\langle w_k, w_k\rangle = \langle\tau(((B_2)^k)^*(B_2)^k)w_0, w_0\rangle, \qquad \langle w_i, w_j\rangle = 0, \quad i \neq j.$$

Observe that

$$\begin{aligned}\langle w_k, w_k\rangle &= \langle\tau(((B_2)^k)^*(B_2)^k)w_0, w_0\rangle\\ &= (-1)^k\overline{c_2}^k q^{3\binom{k}{2}}\langle\tau(K^k(B_1)^k(B_2)^k)w_0, w_0\rangle = (-1)^k\overline{c_2}^k q^{3\binom{k}{2}}\langle\tau(K^k)w_0, w_0\rangle\prod_{i=1}^k b_i\\ &= \frac{q^{3\binom{k}{2}-k(2n-1)}}{(1-q^2)^k}[k]_q!\,(q^{2n};q^{-2})_k\,(-c_2c_1^{-1}\kappa^2q^{2n-1};q^2)_k\,\langle w_0, w_0\rangle. \end{aligned} \tag{3.7}$$

Since $q^3c_2c_1^{-1} = c_1\overline{c_2}c_2c_1^{-1} = |c_2|^2 > 0$, it follows that $c_2c_1^{-1} > 0$ and thus (3.7) is positive. Therefore $\langle\cdot,\cdot\rangle$ is a positive definite bilinear form. Moreover, $\langle\tau(X)w_i, w_j\rangle = \langle w_i, \tau(X^*)w_j\rangle$ for all $X \in \mathcal{B}$. This follows from a straightforward verification on the generators of $\mathcal{B}$. □

Remark 3.8 Let $\kappa \in \mathbb{R}^\times$ and $n \in \mathbb{N}$. Let $(w_i)_{i=0}^n$ be the orthogonal basis for $W_{(\mu,n)}$ as in Corollary 3.3. We define an orthonormal basis $(\widetilde{w}_i)_{i=0}^n$ by $\widetilde{w}_i = w_i/||w_i||$. The actions of C_1, C_2 and K on the orthonormal basis are the same. For B_1 and B_2 we have

$$\begin{aligned}\tau_{(\kappa,n)}(B_1)\widetilde{w}_i &= -c_1\kappa^{-1}q^{-2i-n+1}\sqrt{\frac{(1-q^{2i})}{(1-q^2)}\frac{(1-q^{2n-2i+2})}{(1-q^2)}(q+c_2c_1^{-1}\kappa^2q^{2n+2i})}\,\widetilde{w}_{i-1},\\ \tau_{(\kappa,n)}(B_2)\widetilde{w}_i &= q^{i-n+1}\sqrt{\frac{(1-q^{2i+2})}{(1-q^2)}\frac{(1-q^{2n-2i})}{(1-q^2)}(q+c_2c_1^{-1}\kappa^2q^{2n+2i+2})}\,\widetilde{w}_{i+1}.\end{aligned}$$

4 The Branching Rule

In this section we prove the main theorem of the paper. We fix a coideal subalgebra $\mathcal{B}$ and show that any finite-dimensional representation of $\mathcal{U}_q(\mathfrak{su}(3))$ restricted to $\mathcal{B}$ decomposes multiplicity free as finite-dimensional representations of $\mathcal{B}$ and we characterise the $\mathcal{B}$-representations that occur in this decomposition. In case $\mathcal{B}$ is $*$-invariant, every finite-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{su}(3))$ restricted to $\mathcal{B}$ obviously decomposes into finite-dimensional irreducible representations. This fact is also noted by Letzter [17, Theorem 3.3].

Theorem 4.1 *Let $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2 \in P^+$ and fix the finite-dimensional irreducible unitary representation π_λ of $\mathcal{U}_q(\mathfrak{su}(3))$ on the vector space V_λ. Let $\mathcal{B}$ be a coideal subalgebra with $c_2c_1^{-1} \notin -q^{2\lambda_1+2\lambda_2+1-\mathbb{N}}$. The representation π_λ restricted to $\mathcal{B}$ decomposes multiplicity free into irreducible representations;*

$$\pi_\lambda|_{\mathcal{B}} \simeq \bigoplus_{(\kappa,n)} \tau_{(\kappa,n)}, \qquad V_\lambda = \bigoplus_{(\kappa,n)} W_{(\kappa,n)},$$

where the sum is taken over $(\kappa, n) = (q^{\lambda_1-\lambda_2-3i}, i + x)$, with $0 \le i \le \lambda_1$ and $0 \le x \le \lambda_2$.

The proof of Theorem 4.1 will be carried out in the next subsections. If $(\tau_{(\kappa,n)}, W_{(\kappa,n)})$ is a representation of $\mathcal{B}$ that occurs in the representation π_λ upon restriction to $\mathcal{B}$ then, up to a scalar, a highest weight vector $w_0^{(\kappa,n)}$ for $\tau_{(\kappa,n)}$ is completely determined by the highest weight κ and the eigenvalue η_1, see Remark 3.5. Hence, highest weight vectors for $\mathcal{B}$-representations in V_λ are eigenvectors of $\pi_\lambda(C_1)$ belonging to the kernel of $\pi_\lambda(B_1)$. In Section 4.1 we determine the kernel of $\pi_\lambda(B_1)$.

Remark 4.2 Observe that the Serre relations (3.2) for $\mathcal{B}$ imply that the kernel of $\pi_\lambda(B_1)$ is invariant under the action of B_1B_2 and thus under the action of C_1.

In Section 4.2 we diagonalize the restriction of $\pi_\lambda(C_1)$ to $\ker(\pi_\lambda(B_1))$. In most of the proofs we denote $\pi_\lambda(X)$, $X \in \mathcal{U}_q(\mathfrak{su}(3))$, with X.

Remark 4.3 The restriction on c_1 and c_2 in Theorem 4.1 is assumed in order to ensure the complete reducibility of π_λ upon restriction to $\mathcal{B}$. This is not always true for the excluded values of c_1 and c_2. For example let $\lambda = \varpi_1$. Then V_λ is a three dimensional vector space. Mudrov's basis in Theorem 2.3 is given by

$$\mathscr{B} = \{v_\lambda,\ F_1\, v_\lambda,\ F_2F_1\, v_\lambda\}.$$

In this basis, the operator C_1 is given by the 3×3 matrix

$$C_1 = \begin{pmatrix} \frac{c_1q^2+c_1-qc_2)}{q(q^2-1)} & 0 & -c_1c_2 \\ 0 & \frac{x_1q^4+c_1-qc_2}{q^2-1} & 0 \\ -q & 0 & \frac{c_1q^4+c_1-q^3c_2}{q(q^2-1)} \end{pmatrix}.$$

The eigenvectors of C_1 are (multiples of) the vectors

$$\rho_1 = \begin{pmatrix} c_1 \\ 0 \\ 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} -c_2/q \\ 0 \\ 1 \end{pmatrix}.$$

If $c_1 \neq -c_2/q$, then V_λ decomposes as a sum of a two-dimensional and a one-dimensional irreducible representations of W:

$$V_\lambda = W_{(q,0)} \oplus W_{(q^{-2},1)},$$

where $W_{(q,0)} = \langle\{\rho_1\}\rangle$ and $W_{(q^{-2},1)} = \langle\{\rho_2, \rho_3\}\rangle$. Moreover, the highest weight vectors of $W_{(q,0)}$ and $W_{(q^{-2},1)}$ are ρ_1 and ρ_2 respectively. If we let $c_1 = -c_2/q$ then the matrix C_1 degenerates into a non-diagonalizable matrix. The only eigenvectors are the multiples of ρ_2 and ρ_3 and therefore, although $W_{(q^{-2},1)}$ is a $\mathcal{B}$-invariant subspace of V_λ, there is no one-dimensional $\mathcal{B}$-invariant subspace in V_λ.

4.1 The Kernel of B_1

The goal of this subsection is to describe the structure of the kernel of $\pi_\lambda(B_1)$ by introducing a particular basis. For each $i = 0, \dots, \lambda_1$, we introduce the following subspaces of V_λ:

$$U_i = \langle \mathscr{B}_i \rangle, \qquad \mathscr{B}_i = \{F_2^k \hat{F}_3^l F_1^{i+k} v_\lambda : 0 \le l \le \lambda_2,\ 0 \le k \le \lambda_1 - i\}. \tag{4.1}$$

Note $U_i = \ker(K - q^{\lambda_1 - \lambda_2 + 3i})$, so that the U_i are mutually orthogonal subspaces of V_λ. By (3.4) we have $B_1 : U_i \to U_{i+1}$, $B_2 : U_i \to U_{i-1}$ and by Proposition 2.4 we have

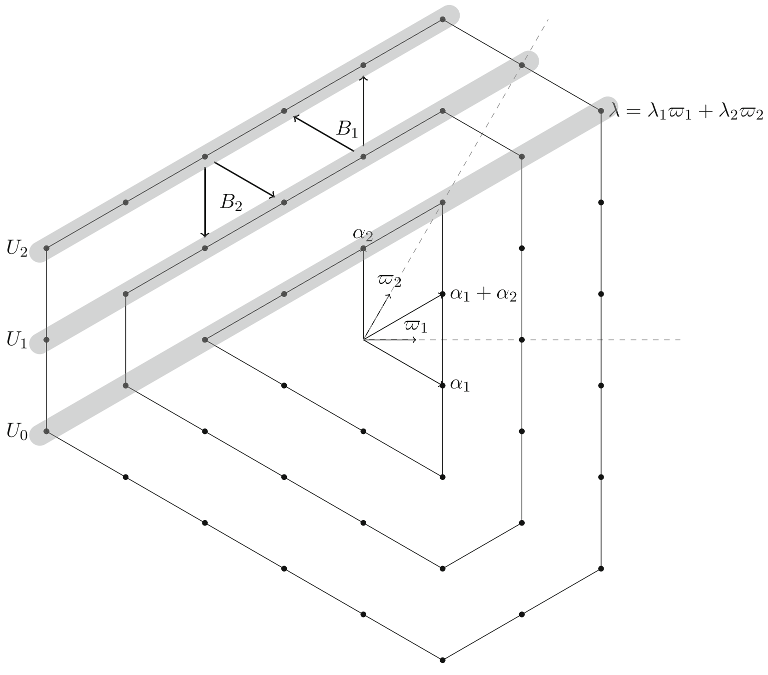

Fig. 1 Weight diagram for the weight $\lambda = 2\varpi_1 + 5\varpi_2$. The multiplicities are as in the Lie algebra case. In particular, dim $V_\lambda(\nu) = 1$ for ν in the outer hexagon, dim $V_\lambda(\nu) = 2$ in the inner hexagon and dim $V_\lambda(\nu) = 3$ in the remaining weight spaces. The subspaces U_i defined in (4.1) are spanned by the basis vectors in the weight spaces indicated in the gray lines. Each vector in the weight space $V_\lambda(\nu)$ is an eigenvector for K, where the eigenvalue is constant along the line parallel to $\alpha_1 + \alpha_2$. In particular the U_i's are the eigenspaces for K for the eigenvalue $q^{\lambda_1 - \lambda_2 + 3i}$

$F_1, E_2 : U_i \to U_{i+1}$ and $F_2, E_1 : U_{i+1} \to U_i$. This is shown in Fig. 1 for the highest weight $\lambda = 2\varpi_1 + 5\varpi_2$.

Remark 4.4 For each $i = 0, \dots, \lambda_1$, the basis $\mathscr{B}_i$ consists on $\lambda_1 - i + 1$ layers of $\lambda_2 + 1$ vectors. More precisely, for $k = 0, \dots, \lambda_1 - i$, the k-th layer is given by the vectors

$$F_2^k \hat{F}_3^l F_1^{k+i} v_\lambda, \qquad l = 0, \dots, \lambda_2.$$

This structure is indicated in the Fig. 2 for the representation $\lambda = 2\varpi_1 + 5\varpi_2$. The layers appear as circled numbers.

Remark 4.5 The dimension of U_i is $(\lambda_2 + 1)(\lambda_1 - i + 1)$. Therefore, the dimension of $\ker(B_1)|_{U_i}$ is, at least, $\lambda_2 + 1$. In particular, $U_{\lambda_1} \subset \ker(B_1)$.

Proposition 4.6 *The kernel of* $\pi_\lambda(B_1)|_{U_i}$ *has dimension* $\lambda_2 + 1$. *Moreover, a basis of* $\ker \pi_\lambda(B_1)|_{U_i}$ *is given by* $(u_n^i)_{n=0}^{\lambda_2}$, *where*

$$u_n^i = \sum_{k=0}^{\lambda_1 - i} \sum_{l=0}^{\lambda_2} \gamma_{k,l}^n \, F_2^k \hat{F}_3^l F_1^{k+i} \, v_\lambda,$$

and the coefficients $\gamma_{k,l}^n$ *are given by the recurrence relation*

$$\begin{aligned} a_k(l, k+i)\, \gamma_{k,l}^n + b_{k+1}(l-1, k+i+1)\, \gamma_{k+1,l-1}^n \\ -c_1\, q^{l+2i+k+1-\lambda_1}\, \eta_{k+1}(l, k+i+1)\, \gamma_{k+1,l}^n = 0, \end{aligned}$$

for $k = 1, \dots, \lambda_1 - i - 1$, $l = 0, \dots, \lambda_2$, *with initial values* $\gamma_{\lambda_1 - i, l}^n = \delta_{n,l}$.

Proof Let $u = \sum_{k=0}^{\lambda_1 - i} \sum_{l=0}^{\lambda_2} \gamma_{k,l}\, F_2^k \hat{F}_3^l F_1^{k+i}\, v_\lambda$ be a vector in the kernel of B_1. Then

$$\begin{aligned} B_1 u &= \sum_{k=0}^{\lambda_1 - i} \sum_{l=0}^{\lambda_2} \gamma_{k,l} \, (F_1 F_2^k \hat{F}_3^l F_1^{k+i}\, v_\lambda - c_1 E_2 K_1^{-1} F_2^k \hat{F}_3^l F_1^{k+i}\, v_\lambda) \\ &= \sum_{k=0}^{\lambda_1 - i} \sum_{l=0}^{\lambda_2} \gamma_{k,l} \, (a_k(l, k+i)\, F_2^k \hat{F}_3^l F_1^{k+i+1}\, v_\lambda + b_k(l, k+i)\, F_2^{k-1} \hat{F}_3^{l+1} F_1^{k+i}\, v_\lambda \\ &\qquad - c_1 q^{l+2i+k-\lambda_1} \eta_k(l, k+i)\, F_2^{k-1} \hat{F}_3^l F_1^{k+i}\, v_\lambda) \\ &= \sum_{k=0}^{\lambda_1 - i} \sum_{l=0}^{\lambda_2} (a_k(l, k+i)\, \gamma_{k,l} + b_{k+1}(l-1, k+i+1)\, \gamma_{k+1,l-1} \\ &\qquad - c_1\, q^{l+2i+k+1-\lambda_1}\, \eta_{k+1}(l, k+i+1) \gamma_{k+1,l})\, F_2^k \hat{F}_3^l F_1^{k+i+1}\, v_\lambda. \end{aligned}$$

Since the elements $F_2^k \hat{F}_3^l F_1^{k+i+1}$, $0 \le k \le \lambda_1 - i$, $0 \le l \le \lambda_2$, are linearly independent it follows that the coefficients $\gamma_{k,l}$ satisfy the following recurrence relation.

$$a_k(l, k+i)\, \gamma_{k,l} + b_{k+1}(l-1, k+i+1)\, \gamma_{k+1,l-1} - c_1\, q^{l+2i+k+1-\lambda_1}\, \eta_{k+1}(l, k+i+1)\, \gamma_{k+1,l} = 0. \tag{4.2}$$

For each $n = 0, 1, \dots, \lambda_2$, if we set $\gamma_{\lambda_1 - i, l}^n = \delta_{n,l}$, then (4.2) determines uniquely a vector u_n in the kernel of B_1. The vectors u_n are clearly linearly independent and span the kernel of B_1 restricted to U_i. This completes the proof of the proposition. □

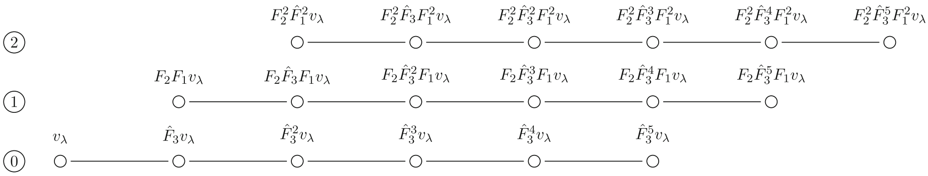

Fig. 2 Structure of the basis of U_0 for the representation π_λ with $\lambda = 2\varpi_1 + 5\varpi_2$ as in Fig. 1. The circled numbers indicate the layers of the basis. Note that vertically aligned basis elements span the weight space $V_\lambda(\lambda - r(\alpha_1 + \alpha_2))$ for $r = 0$ (*left*) to $r = \lambda_1 + \lambda_2 = 7$ (*right*)

Remark 4.7 According to the layer structure of $\mathscr{B}_i$ described in Remark 4.4, the vector u_n^i has a single non-zero contribution from the vectors of the upper layer, namely from $F_2^{\lambda_1-i}\hat{F}_3^n F_1^{\lambda_1}$, and two contributions from the one but upper layer. Therefore, we have

$$u_n^i = F_2^{\lambda_1-i}\hat{F}_3^n F_1^{\lambda_1} v_\lambda + \gamma^n_{\lambda_1-i-1,n} F_2^{\lambda_1-i-1}\hat{F}_3^n F_1^{\lambda_1-1} v_\lambda + \gamma^n_{\lambda_1-i-1,n+1} F_2^{\lambda_1-i-1}\hat{F}_3^{n-1} F_1^{\lambda_1-1} v_\lambda + \sum_{k=0}^{\lambda_1-i-2}\sum_{l=0}^{\lambda_2}\gamma^n_{k,l} F_2^k \hat{F}_3^l F_1^{i+k} v_\lambda. \tag{4.3}$$

The coefficients $\gamma^n_{\lambda_1-i-1,n}$ and $\gamma^n_{\lambda_1-i-1,n+1}$ corresponding to the vectors of the one but last layer are given by

$$\begin{aligned} \gamma^n_{\lambda_1-i-1,n} &= \frac{c_1 q^{n+i}(q^{\lambda_1-i}-q^{-\lambda_1+i})(q^{\lambda_2+\lambda_1-n}-q^{-\lambda_2-\lambda_1+n})}{(q-q^{-1})^2}, \\ \gamma^n_{\lambda_1-i-1,n+1} &= -\frac{(q^{\lambda_1-i}-q^{-\lambda_1+i})(q^{\lambda_2+\lambda_1-n-1}-q^{-\lambda_2-\lambda_1+n+1})}{(q^{\lambda_2+\lambda_1+1-n}-q^{-\lambda_2-\lambda_1-1+n})(q^{\lambda_2+i-n}-q^{-\lambda_2-i+n})}. \end{aligned} \tag{4.4}$$

The structure of the vectors u_n^i for U_{λ_1-2} is depicted in Fig. 3.

Remark 4.8 The basis $\{u_n^i\}_n^i$ of the kernel of $\pi_\lambda(B_1)$ is not an orthogonal basis. In fact, it follows from Remark 4.7 that

$$\begin{aligned} u_0^{\lambda_1-1} &= F_2^{\lambda_1-1}F_1^{\lambda_1} v_\lambda + \gamma^0_{\lambda_1-2,0} F_2^{\lambda_1-2}F_1^{\lambda_1-1} v_\lambda, \\ u_1^{\lambda_1-1} &= F_2^{\lambda_1-1}\hat{F}_3F_1^{\lambda_1} v_\lambda + \gamma^1_{\lambda_1-2,1} F_2^{\lambda_1-2}\hat{F}_3F_1^{\lambda_1-1} v_\lambda + \gamma^1_{\lambda_1-2,2} F_2^{\lambda_1-2}F_1^{\lambda_1-1} v_\lambda, \end{aligned}$$

and therefore

$$\langle u_0^{\lambda_1-1}, u_1^{\lambda_1-1}\rangle = \gamma^0_{\lambda_1-2,0}\gamma^1_{\lambda_1-2,2} H_{\lambda_1-2,0,\lambda_1-1} \neq 0,$$

using the explicit expressions (4.4).

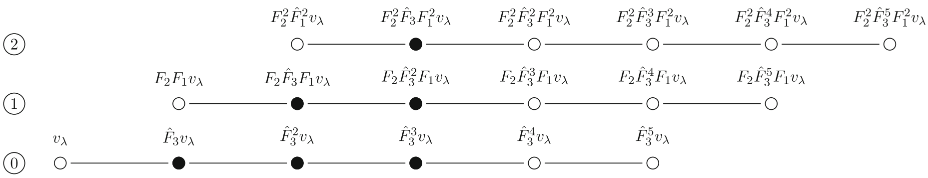

Fig. 3 Structure of the basis $(u_n^0)_n$ of $\ker(B_1|_{U_0})$ for the representation $\lambda = 2\varpi_1 + 5\varpi_2$ as in Fig. 1. The black circles indicate the terms that contribute to the expression of the element $u_1^0 = F_2^2\hat{F}_3F_1^2 v_\lambda + \cdots$

4.2 The Action of C_1

In Remark 4.2 we observed that the kernel of B_1 is stable under the action of C_1. Furthermore for each $i = 0, \ldots, \lambda_1$, U_i is stable under C_1. The goal of this subsection is to compute the action of C_1 in the basis of $\ker \pi_\lambda(B_1)$ given in Proposition 4.6.

Lemma 4.9 *In the basis $\mathscr{B}$ of Theorem 2.3 we have*

$$\begin{aligned}
F_1F_2F_2^k\hat{F}_3^lF_1^{k+i}\,v_\lambda &= a_{k+1}(l,k+i)\,F_2^{k+1}\hat{F}_3^lF^{k+i+1}\,v_\lambda + b_{k+1}(l,k+i)\,F_2^k\hat{F}_3^{l+1}F_1^{k+i}\,v_\lambda,\\
E_2F_2F_2^k\hat{F}_3^lF_1^{k+i}\,v_\lambda &= \eta_{k+1}(l,k+i)\,F_2^k\hat{F}_3^l\,F_1^{k+i}\,v_\lambda,\\
F_1E_1F_2^k\hat{F}_3^lF_1^{k+i}\,v_\lambda &= \alpha_k(l,k+i)\,a_k(l,k+i-1)\,F_2^k\hat{F}_3^lF_1^{k+i}\,v_\lambda\\
&\quad+\alpha_k(l,k+i)\,b_k(l,k+i-1)\,F_2^{k-1}\hat{F}_3^{l+1}F_1^{k+i-1}\,v_\lambda\\
&\quad+\beta_k(l,k+i)\,a_{k+1}(l-1,k+i)F_2^{k+1}\hat{F}_3^{l-1}F_1^{k+i+1}\,v_\lambda\\
&\quad+\beta_k(l,k+i)\,b_{k+1}(l-1,k+i)F_2^k\hat{F}_3^lF_1^{k+i}\,v_\lambda,\\
E_2E_1F_2^k\hat{F}_3^lF_1^{k+i} &= \alpha_k(l,k+i)\,\eta_k(l,k+i-1)\,F_2^{k-1}\hat{F}_3^lF_1^{k+i-1}\,v_\lambda\\
&\quad+\beta_k(l,k+i)\,\eta_{k+1}(l-1,k+i)\,F_2^k\hat{F}_3^{l-1}F_1^{k+i}\,v_\lambda,\\
KF_2^{\lambda_1-i}\hat{F}_3^lF_1^{\lambda_1}\,v_\lambda &= q^{\lambda_1-\lambda_2-3i}\,F_2^{\lambda_1-i}\hat{F}_3^lF_1^{\lambda_1}\,v_\lambda,\\
K^{-1}F_2^{\lambda_1-i}\hat{F}_3^lF_1^{\lambda_1}\,v_\lambda &= q^{\lambda_2-\lambda_1+3i}\,F_2^{\lambda_1-i}\hat{F}_3^lF_1^{\lambda_1}\,v_\lambda.
\end{aligned}$$

Proof The lemma is a direct consequence of Proposition 2.4. □

Since K acts as a multiple of the identity on each U_i, it suffices to determine the action of B_1B_2 on U_i.

Lemma 4.10 *For $i \in 0, \ldots, \lambda_1$, in the basis $(u_n^i)_n$ of* $\ker(B_1)$*, we have*

$$B_1B_2\,u_n^i = A(n)u_{n+1}^i + B(n)u_n^i + C(n)u_{n-1}^i, \qquad n = 0, \ldots, \lambda_2,$$

where

$$\begin{aligned}
A(n) &= \frac{q^{\lambda_2+i-n}(1-q^2)(1-q^{2\lambda_1+2\lambda_2-2n})}{(1-q^{2\lambda_2+2\lambda_1-2n+2})(1-q^{2\lambda_2+2i-2n})},\\
B(n) &= -c_1\frac{q^{2n+i-\lambda_1-\lambda_2}\,(1-q^{2\lambda_2-2n+2i})}{(1-q^2)} + \frac{c_2\,q^{\lambda_1-\lambda_2+2n-i+1}(1-q^{-2n-2i})}{(1-q^2)},\\
C(n) &= \frac{c_1c_2\,q^{3n-3\lambda_2-i-2}(1-q^{2n})(1-q^{2\lambda_2-2n+2})(1-q^{2\lambda_1+2\lambda_2-2n+4})(1-q^{2\lambda_2+2i+2-2n})}{(1-q^2)^3(1-q^{2\lambda_2+2\lambda_1+2-2n})}.
\end{aligned}$$

Proof Since U_i is stable under B_1B_2 and $(u_n^i)_n$ is a basis of $\ker(B_1|_{U_i})$, we have

$$B_1B_2\,u_n^i = \sum_{j=0}^{\lambda_2} \nu_j\,u_j^i,$$

for certain coefficients ν_j. Since $\mathscr{B}_i$ is an orthogonal basis and u_n^i has a single contribution from the vectors in the upper layer of $\mathscr{B}_i$, see Remark 4.7, we obtain that

$$\langle B_1 B_2 u_n^i,\ F_2^{\lambda_1-i}\ \hat{F}_3^s\ F_1^{\lambda_1}\ v_\lambda\rangle = \sum_{j=0}^{\lambda_2} \nu_j \langle u_j^i,\ F_2^{\lambda_1-i}\ \hat{F}_3^s\ F_1^{\lambda_1}\ v_\lambda\rangle = \nu_s H^2_{\lambda_1-i,s,\lambda_1}.$$

On the other hand, from (3.1) we have

$$\begin{aligned} B_1 B_2 F_2^k \hat{F}_3^l F_1^{k+i}\ v_\lambda &= F_1 F_2 F_2^k \hat{F}_3^l F_1^{k+i}\ v_\lambda - c_1 q^{l+k+2i-1-\lambda_1} E_2 F_2 F_2^k \hat{F}_3^l F_1^{k+i}\ v_\lambda \\ &-c_2 q^{k+l-i-\lambda_2} F_1 E_1 F_2^k \hat{F}_3^l F_1^{k+i}\ v_\lambda + c_1 c_2\, q^{2l+2k+i-\lambda_1-\lambda_2-2} E_2 E_1 F_2^k \hat{F}_3^l F_1^{k+i}\ v_\lambda. \end{aligned} \tag{4.5}$$

Applying Lemma 4.9 to (4.5), we verify that the action of $B_1 B_2$ on the vector of the k-th layer $F_2^k \hat{F}_3^l F_1^{k+i}\ v_\lambda$ has contributions from the $(k-1)$-th, k-th and $(k+1)$-th layer. Hence, Remark 4.7 implies

$$\begin{aligned} \langle B_1 B_2 u_n^i,\ F_2^{\lambda_1-i} \hat{F}_3^s\ F_1^{\lambda_1}\ v_\lambda\rangle = &\langle B_1 B_2\ F_2^{\lambda_1-i} \hat{F}_3^n F_1^{\lambda_1}\ v_\lambda,\ F_2^{\lambda_1-i}\ \hat{F}_3^s\ F_1^{\lambda_1}\ v_\lambda\rangle \\ &+\gamma^n_{\lambda_1-i-1,n} \langle B_1 B_2\ F_2^{\lambda_1-i-1} \hat{F}_3^n F_1^{\lambda_1-1}\ v_\lambda,\ F_2^{\lambda_1-i}\ \hat{F}_3^s\ F_1^{\lambda_1}\ v_\lambda\rangle \\ &+\gamma^n_{\lambda_1-i-1,n+1} \langle B_1 B_2\ F_2^{\lambda_1-i-1} \hat{F}_3^{n-1} F_1^{\lambda_1-1}\ v_\lambda,\ F_2^{\lambda_1-i} \hat{F}_3^s F_1^{\lambda_1} v_\lambda\rangle. \end{aligned} \tag{4.6}$$

From Lemma 4.9 we obtain that (4.6) is zero unless $s = n-1, n, n+1$. Moreover, we have

$$\begin{aligned} &\langle B_1 B_2 u_n^i,\ F_2^{\lambda_1-i}\ \hat{F}_3^{n+1}\ F_1^{\lambda_1}\ v_\lambda\rangle = [b_{\lambda_1-i+1}(n,\lambda_1) + \gamma^n_{\lambda_1-i-1,n+1}\, a_{\lambda_1-i}(n+1, \lambda_1-1)] H^2_{\lambda_1-i,n+1,\lambda_1}, \\ &\langle B_1 B_2 u_n^i,\ F_2^{\lambda_1-i}\ \hat{F}_3^n\ F_1^{\lambda_1}\ v_\lambda\rangle = [-c_1\, q^{n+i-1}\, \eta_{\lambda_1-i+1}(l,\lambda_1) \\ &\quad -c_2\, q^{\lambda_1+n-2i-\lambda_2}\, \alpha_{\lambda_1-i}(n,\lambda_1)\, a_{\lambda_1-i}(n,\lambda_1-1) \\ &\qquad -c_2\, q^{\lambda_1-2i+n-\lambda_2}\, \beta_{\lambda_1-i}(n,\lambda_1)\, b_{\lambda_1-i+1}(n-1,\lambda_1) + \gamma^n_{\lambda_1-i-1,n}\, a_{\lambda_1-i}(n,\lambda_1-1) \\ &\qquad\quad -c_2\, q^{\lambda_1+n-2i-\lambda_2} \gamma^n_{\lambda_1-i-1,n+1}\, \beta_{\lambda_1-i-1}(n+1,\lambda_1-1)\, a_{\lambda_1-i}(n,\lambda_1-1)] H^2_{\lambda_1-i,n,\lambda_1}, \\ &\langle B_1 B_2 u_n^i,\ F_2^{\lambda_1-i}\ \hat{F}_3^{n-1}\ F_1^{\lambda_1}\ v_\lambda\rangle = [c_1 c_2\, q^{\lambda_1-\lambda_2+2n-i-2}\, \beta_{\lambda_1-i}(n,\lambda_1)\, \eta_{\lambda_1-i+1}(n-1,\lambda_1) \\ &\quad -c_2 q^{\lambda_1-\lambda_2+n-2i-1}\, \gamma^n_{\lambda_1-i-1,n}\, \beta_{\lambda_1-i-1}(n,\lambda_1-1)\, a_{\lambda_1-i}(n-1,\lambda_1-1)] H^2_{\lambda_1-i,n-1,\lambda_1}. \end{aligned}$$

Now the lemma follows from Proposition 2.4 and (4.4). □

Lemma 4.11 *For $i \in 0,\dots,\lambda_1$, in the basis $(u_n^i)_n$ of* $\ker(B_1)$*, we have*

$$C_1\, u_n^i = A(n) u_{n+1}^i + (B(n)+D)\, u_n^i + C(n) u_{n-1}^i, \quad D = -c_2 \frac{q^{\lambda_1-\lambda_2-3i}}{q-q^{-1}} + c_1 \frac{q^{\lambda_2-\lambda_1+3i}(q+q^{-1})}{q-q^{-1}}.$$

Proof Lemma 4.10, (3.3) and K acting as a multiple of the identity give the result. □

We are now ready to find the eigenvectors of C_1 restricted to $\ker(B_1)|_{U_i}$. We will describe these eigenvectors as a linear combination of the vectors u_n^i with explicit coefficients given in terms of dual q-Krawtchouk polynomials. For $N \in \mathbb{N}$ and $n = 0, 1, \dots, N$, the dual q-Krawtchouk polynomials are given explicitly by

$$K_n(\lambda(x); c, N|q) = \frac{(q^{x-N}; q)_n}{(q^{-N}; q)_n q^{nx}}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-x} \\ q^{N-x-n+1} \end{matrix} \middle| q, cq^{x+1} \right),$$

where $\lambda(x) = q^{-x} + cq^{x-N}$, see [10, (3.17.1)]. We follow the standard notation of [7] for basic hypergeometric series, namely

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| q, z\right) = \sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(c; q)_k} \frac{z^k}{(q; q)_k}. $$

The polynomials

$$ r_l(\lambda(x)) = (q^{-2N}; q)_l\, K_l(\lambda(x); c, N|q^2), \tag{4.7} $$

satisfy the three term recurrence relation

$$ x\, r_l(x) = r_{l+1}(x) + (1+c)q^{2l-2N}\, r_l(x) + c\, q^{-2N}(1-q^{2l})(1-q^{2l-2N-2})\, r_{l-1}(x). \tag{4.8} $$

Proposition 4.12 *For $i = 0, \dots, \lambda_1$, the set $\{\psi_x^i\}_{x=0}^{\lambda_2}$ where*

$$ \psi_x^i = \sum_{l=0}^{\lambda_2} \frac{(-1)^l\, c_2^{-l} q^{-l(\lambda_1+4)+l(l+1)/2}\,(1-q^{2l})(q^{-2\lambda_2-2\lambda_1}; q^2)_l}{(q^2, q^{-2\lambda_2-2\lambda_1-2}, q^{-2\lambda_2-2i}; q^2)_l} K_l(\lambda(x), -c_1^{-1}c_2 q^{2\lambda_1-2i+1}, \lambda_2, q^2)\, u_l^i, $$

is a basis of eigenvectors of C_1 restricted to $\ker(B_1)|_{U_i}$. *The eigenvalue of ψ_x^i is*

$$ \eta_1 = \frac{c_1\, \kappa^{-1}\, q(1+q^{-2n-2}) - c_2\, \kappa\, q^{2n}}{q - q^{-1}}, $$

for $\kappa = q^{\lambda_1-3i-\lambda_2}$ and $n = x + i$.

Remark 4.13 As we pointed out in Remark 4.8, the basis $(u_n^i)_n$ is not orthogonal. Still the operator C_1 acts tridiagonally. Moreover, if $\mathcal{B}$ is $*$-invariant then the basis $\{\psi_x^i\}_{x=0}^{\lambda_2}$ in Proposition 4.12 is orthogonal although, because of the non-orthogonality of $(u_n^i)_n$, this does not follow directly from the orthogonality of the dual q-Krawtchouk polynomials.

Proof Assume there exist polynomials $p_n(x)$ such that $v = \sum_{l=0}^{\lambda_2} p_l(x)\, u_l^i$ is an eigenvector of C_1 with eigenvalue η_1, i.e. $C_1\, v = \eta_1\, v$. From Lemma 4.11 we have

$$ C_1\, v = \sum_{l=0}^{\lambda_2} p_l(x)(A(l)u_{l+1}^i + (B(l) + D)\, u_l^i + C(l)u_{l-1}^i) = \sum_{l=0}^{\lambda_2} \eta_1\, p_l(x)\, u_l^i. $$

Since $(u_l^i)_l$ is a basis of $\ker(B_1)|_{U_i}$ the vectors u_l^i are linearly independent and hence the polynomials p_l satisfy the following three term recurrence relation

$$ \eta_1\, p_l(x) = C(l+1)p_{l+1}(x) + (B(l) + D)\, p_l(x) + A(l-1)p_{l-1}(x). $$

If k_l is the leading coefficient of p_l, then $P_l = k_l^{-1} p_l$ is a sequence of monic polynomials satisfying the recurrence relation

$$ \eta_1\, P_l(x) = P_{l+1}(x) + (B(l) + D)\, P_l(x) + C(l)A(l-1)P_{l-1}(x), \tag{4.9} $$

where

$$ \begin{aligned} B(l) + D \;&=\; -\frac{c_1\, q^{2l+i-\lambda_1-\lambda_2}(1 - c_1^{-1}c_2\, q^{2\lambda_1-2i+1})}{(1-q^2)} - \frac{c_1\, q^{3i-\lambda_1+\lambda_2+2}}{(1-q^2)}, \\ C(l)A(l-1) \;&=\; -\frac{c_1c_2\, q\,(1-q^{2l})(1-q^{2l-2\lambda_2-2})}{(1-q^2)^2}, \end{aligned} $$

using Lemma 4.10 and Lemma 4.11. We will identify the polynomials P_l with the dual q-Krawtchouk polynomials. If we let

$$c = -c_1^{-1} c_2\, q^{2\lambda_1 - 2i + 1}, \quad N = \lambda_2,$$

the recurrence relation (4.8) is given by

$$\begin{aligned} x\, r_l(x) \; &= \; r_{l+1}(x) + (1 + c_1^{-1} c_2 q^{2\lambda_1 - 2i+1}) q^{2l - 2\lambda_2}\, r_l(x) \\ &\quad + c_1^{-1} c_2 q^{2\lambda_1 - 2\lambda_2 - 2i + 1}(1 - q^{2l})(1 - q^{2l - 2\lambda_2 - 2})\, r_{l-1}(x). \end{aligned}$$

If we let $\tilde{r}_l(x) = a^{-l}\, r_l(ax)$ with $a = -c_1^{-1}\, q^{\lambda_1 - \lambda_2 - i}(1 - q^2)$, by a straightforward computation we obtain

$$\left(x - \frac{c_1\, q^{3i - \lambda_1 + \lambda_2 + 2}}{(1 - q^2)} \right) \tilde{r}_l(x) = \tilde{r}_{l+1}(x) + (B(l) + D)\, \tilde{r}_l(x) + C(l) A(l-1)\, \tilde{r}_{l-1}(x). \quad (4.10)$$

If we evaluate (4.10) in $\lambda(x) a^{-1}$, the eigenvalue is given by

$$\frac{\lambda(x)}{a} - \frac{c_1\, q^{3i - \lambda_1 + \lambda_2 + 2}}{(1 - q^2)} = \frac{c_1\, q^{3i - \lambda_1 + \lambda_2 + 2}(1 + q^{-2x - 2i - 2}) + c_2\, q^{\lambda_1 - \lambda_2 - i + 2x}}{q - q^{-1}}.$$

Therefore the polynomials $P_l(x) = \tilde{r}(\lambda(x) a^{-1}) = a^{-l}\, r_l(\lambda(x))$ satisfy the recurrence (4.9) with eigenvalue

$$\eta_1 = \frac{c_1\, \kappa^{-1}\, q(1 + q^{-2n-2}) - c_2\, \kappa\, q^{2n}}{q - q^{-1}},$$

with $\kappa = q^{\lambda_1 - 3i - \lambda_2}$ and $n = x + i$, for $x = 0, \ldots, \lambda_2$. Finally, $p_l(x) = k_l\, a^{-l} r_l(\lambda(x))$. The explicit expression of p_l follows from (4.7) and Lemma 4.10. $\square$

Proof of Theorem 4.1 From Proposition 4.12 we obtain vectors ψ_x^i for $i = 0, \ldots, \lambda_1$, $x = 0, \ldots, \lambda_2$ such that

$$\pi_\lambda(B_1)\, \psi_x^i = 0, \quad \text{and} \quad C_1\, \psi_x^i = \frac{c_1\, \kappa^{-1}\, q(1 + q^{-2n-2}) - c_2\, \kappa\, q^{2n}}{q - q^{-1}}\, \psi_x^i = \eta_1\, \psi_x^i.$$

where $\kappa = q^{\lambda_1 - 3i - \lambda_2}$ and $n = x + i$, so that ψ_x^i is a highest weight vector. It follows from Corollary 3.6 that the highest weight vector ψ_x^i defines an irreducible representation of $\mathcal{B}$ of dimension $x + i + 1$

$$W_{q^{\lambda_1 - \lambda_2 - 3i}, x+i} = \langle \{ \psi_x^i,\, \pi_\lambda(B_2)\, \psi_x^i,\, \pi_\lambda(B_2)^2\, \psi_x^i,\, \ldots,\, \pi_\lambda(B_2)^{x+i}\, \psi_x^i \} \rangle.$$

Let $W = \oplus_{(\kappa,n)} W_{(\kappa,n)}$ where the sum is taken over $(\kappa, n) = (q^{\lambda_1 - 3i - \lambda_2}, x + i)$ for $i = 0, \ldots, \lambda_1$, $x = 0, \ldots, \lambda_2$. We have that $W \subset V_\lambda$ and

$$\dim W = \sum_{i,x} \dim W_{q^{\lambda_1 - \lambda_2 - 3i}, x+i} = \frac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2) = \dim V_\lambda.$$

Therefore $W = V_\lambda$ and this completes the proof of the theorem. $\square$

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Appendix A. Proof of Theorem 2.3

Lemma A.1 *The following relations hold in* $\mathcal{U}_q(\mathfrak{su}(3))$*:*

(i) $F_2\hat{F}_3[a] = \hat{F}_3[a+1]F_2$,
(ii) $E_1\hat{F}_3[a] = \hat{F}_3[a+1]E_1 + F_2\frac{(q^{a+1}K_1K_2-q^{-a-1}(K_1K_2)^{-1})}{(q-q^{-1})}$,
(iii) $F_2F_3 = qF_3F_2$,
(iv) $(\hat{F}_3[a])^* = q\hat{E}_3[a](K_1K_2)^{-1}$, $F_3^* = qE_3(K_1K_2)^{-1}$,
(v) $\hat{F}_3 = F_3\frac{qK_2-q^{-1}K_2^{-1}}{q-q^{-1}} + qF_2F_1K_2$,
(vi) $E_1F_3 = F_3E_1 + F_2K_1$,

Proof Straightforward verifications using (2.1) and (2.3). □

Corollary A.2 *For* $l \in \mathbb{N}$ *and* $a \in \mathbb{R}$ *we have*

$$E_1\big(\hat{F}_3[a]\big)^l = \big(\hat{F}_3[a+1]\big)^l E_1 + \frac{q^l-q^{-l}}{q-q^{-1}} F_2\big(\hat{F}_3[a]\big)^{l-1}\frac{(q^{a+2-l}K_1K_2 - q^{-a-2+l}(K_1K_2)^{-1})}{(q-q^{-1})}$$

Proof By induction on l using Lemma A.1(ii) and (i). □

Proof of Theorem 2.3 By the PBW-theorem, $F_2^k\hat{F}_3^lF_1^m\,v_\lambda$ for $k,l,m \in \mathbb{N}$ spans V_λ. By Proposition 2.4

$$\begin{aligned} K_1F_2^k\hat{F}_3^lF_1^m\,v_\lambda &= q^{\lambda_1+k-l-2m}F_2^k\hat{F}_3^lF_1^m\,v_\lambda, \\ K_2F_2^k\hat{F}_3^lF_1^m\,v_\lambda &= q^{\lambda_2-2k-l+m}F_2^k\hat{F}_3^lF_1^m\,v_\lambda. \end{aligned} \tag{A.1}$$

Since K_i, $i=1,2$, are self-adjoint, we find that $\langle F_2^k\hat{F}_3^lF_1^m\,v_\lambda, F_2^{k'}\hat{F}_3^{l'}F_1^{m'}\,v_\lambda\rangle = 0$ in case $k-l-2m \neq k'-l'-2m'$ or $-2k-l+m \neq -2k'-l'+m'$. For $k'>k$ we find

$$\begin{aligned} \langle F_2^k\hat{F}_3^lF_1^m\,v_\lambda, F_2^{k'}\hat{F}_3^{l'}F_1^{m'}\,v_\lambda\rangle &= \langle (E_2K_2^{-1})^{k'}F_2^k\hat{F}_3^lF_1^m\,v_\lambda, \hat{F}_3^{l'}F_1^{m'}\,v_\lambda\rangle \\ &= q^{k'(k'+1)}\langle K_2^{-k'}E_2^{k'}F_2^k\hat{F}_3^lF_1^m\,v_\lambda, \hat{F}_3^{l'}F_1^{m'}\,v_\lambda\rangle = 0, \end{aligned} \tag{A.2}$$

since $E_i^{k'}F_i^k \in \mathcal{U}_q(\mathfrak{su}(3))\,E_i^{k'-k}$ for $k,k' \in \mathbb{N}$, $k'>k$, using also Lemma 2.1(ii) for $a=0$, (2.1) and (2.4). Because of the symmetry between k and k', we see that the inner product (A.2) is 0 for $k \neq k'$. With the above remark, we find

$$\langle F_2^k\hat{F}_3^lF_1^m\,v_\lambda, F_2^{k'}\hat{F}_3^{l'}F_1^{m'}\,v_\lambda\rangle = 0$$

in case $k \neq k'$ or $l \neq l'$ or $m \neq m'$.

So it suffices to calculate the norm of the vectors, and see that this is non-zero precisely for the range mentioned. First, using the case $k=k'$ of the first part of (A.2) and that K_2 acts on $E_2^k F_2^k \hat{F}_3^l F_1^m\, v_\lambda$ by the scalar q^{λ_2-l+m}, we find

$$\langle F_2^k \hat{F}_3^l F_1^m\, v_\lambda, F_2^k \hat{F}_3^l F_1^m\, v_\lambda\rangle = q^{k(k+1)-k(\lambda_2-l+m)}\langle E_2^k F_2^k \hat{F}_3^l F_1^m\, v_\lambda, \hat{F}_3^l F_1^m\, v_\lambda\rangle.$$

Now use Lemma 2.2(ii) for $i=2$ and next the commutation relations of Lemma 2.1(ii) and (2.1) to see that the $\mathcal{U}_q(\mathfrak{su}(3))E_2$-part of Lemma 2.2(ii) gives zero contribution. Because of the action of K_2 being diagonal, we find

$$\langle F_2^k \hat{F}_3^l F_1^m\, v_\lambda, F_2^k \hat{F}_3^l F_1^m\, v_\lambda\rangle = \frac{(q^2;q^2)_k}{(1-q^2)^{2k}}(q^{-2(\lambda_2-l+m)};q^2)_k(-1)^k q^{3k}\langle \hat{F}_3^l F_1^m\, v_\lambda, \hat{F}_3^l F_1^m\, v_\lambda\rangle.$$

Next we write

$$\langle \hat{F}_3^l F_1^m\, v_\lambda, \hat{F}_3^l F_1^m\, v_\lambda\rangle = \langle \hat{F}_3^l F_1^m\, v_\lambda, F_1^m \hat{F}_3^l\, v_\lambda\rangle = q^{m(m+1)}q^{-m(\lambda_1-l)}\langle E_1^m \hat{F}_3^l F_1^m\, v_\lambda, \hat{F}_3^l\, v_\lambda\rangle$$

using Lemma 2.1(i), the $*$-structure (2.3), (2.1) and (A.1). Following Mudrov [22, §8] we replace $\hat{F}_3^l$ on the left by F_3^l. First use Lemma A.1(v)

$$\begin{aligned}\langle E_1^m \hat{F}_3^l F_1^m\, v_\lambda, \hat{F}_3^l\, v_\lambda\rangle &= \frac{q^{2+\lambda_2-l+m}-q^{-2-\lambda_2+l-m}}{q-q^{-1}}\langle E_1^m F_3 \hat{F}_3^{l-1} F_1^m\, v_\lambda, \hat{F}_3^l\, v_\lambda\rangle\\ &\quad + q^{2+\lambda_2-l+m}\langle E_1^m F_2 F_1 \hat{F}_3^{l-1} F_1^m\, v_\lambda, \hat{F}_3^l\, v_\lambda\rangle.\end{aligned}$$

In the second term, move F_2 to the left using (2.1), and then the other side so that is essentially an E_2 which we can move through, by Lemma 2.1(ii), to the highest weight vector, and hence gives zero. This we can repeat, since F_2 also q-commutes with F_3 by Lemma A.1(iii). This yields

$$\langle E_1^m \hat{F}_3^l F_1^m\, v_\lambda, \hat{F}_3^l\, v_\lambda\rangle = \frac{(-1)^l q^{l(2+\lambda_2+m)l}q^{-\frac{1}{2}l(l-1)}}{(1-q^2)^l}(q^{-\lambda_2-2-m};q^2)_l\langle E_1^m F_3^l F_1^m\, v_\lambda, \hat{F}_3^l\, v_\lambda\rangle.$$

Using Lemma A.1(vi), and moving F_2 to the other side, where F_2^* kills $\hat{F}_3^l\, v_\lambda$, we see

$$\langle E_1^m F_3^l F_1^m\, v_\lambda, \hat{F}_3^l\, v_\lambda\rangle = (-1)^m q^{-m(m-2)+m\lambda_1}\frac{(q^2;q^2)_m}{(1-q^2)^{2m}}(q^{-2\lambda_1};q^2)_m\langle F_3^l\, v_\lambda, \hat{F}_3^l\, v_\lambda\rangle$$

by Lemma 2.2(ii). Assume $l\geq 1$, so it remains to calculate

$$\begin{aligned}\langle F_3^l\, v_\lambda, \hat{F}_3^l\, v_\lambda\rangle = \langle F_3^{l-1}\, v_\lambda, (F_3)^* \hat{F}_3^l\, v_\lambda\rangle &= q^{1-(\lambda_1+\lambda_2-2l)}\langle F_3^{l-1}\, v_\lambda, (E_2E_1-E_1E_2)\,\hat{F}_3^l\, v_\lambda\rangle\\ &= q^{1-(\lambda_1+\lambda_2-2l)}\langle F_3^{l-1}\, v_\lambda, E_2E_1\hat{F}_3^l\, v_\lambda\rangle\end{aligned}$$

where we use Lemma A.1(iv), the diagonal action of K_i and the fact that the action of E_1E_2 is zero by Lemma 2.1(ii) and (2.4). By Corollary A.2 for $a=0$ and (2.4) we find

$$E_1\hat{F}_3^l\, v_\lambda = \frac{q^l-q^{-l}}{q-q^{-1}}\frac{q^{2+\lambda_1+\lambda_2-l}-q^{-2-\lambda_1-\lambda_2+l}}{q-q^{-1}}F_2\hat{F}_3^{l-1}\, v_\lambda$$

and next applying E_2, using (2.1), (2.4) and Lemma 2.1(ii) we find

$$E_2E_1\hat{F}_3^l\, v_\lambda = \frac{q^l-q^{-l}}{q-q^{-1}}\frac{q^{2+\lambda_1+\lambda_2-l}-q^{-2-\lambda_1-\lambda_2+l}}{q-q^{-1}}\frac{q^{\lambda_2-l+1}-q^{-\lambda_2+l-1}}{q-q^{-1}}\hat{F}_3^{l-1}\, v_\lambda,$$

so that

$$\langle F_3^l\, v_\lambda, \hat{F}_3^l\, v_\lambda\rangle = q^{1-(\lambda_1+\lambda_2-2l)}\frac{q^l - q^{-l}}{q-q^{-1}}$$
$$\times \frac{q^{2+\lambda_1+\lambda_2-l} - q^{-2-\lambda_1-\lambda_2+l}}{q-q^{-1}}\frac{q^{\lambda_2-l+1} - q^{-\lambda_2+l-1}}{q-q^{-1}}\langle F_3^{l-1}\, v_\lambda, \hat{F}_3^{l-1}\, v_\lambda\rangle.$$

Iterating, since we normalize $\langle v_\lambda, v_\lambda\rangle = 1$, we find

$$\langle F_3^l\, v_\lambda, \hat{F}_3^l\, v_\lambda\rangle = q^{l(\lambda_2+7)}q^{-\frac{1}{2}l(l+1)}\frac{(q^2;q^2)_l}{(1-q^2)^{3l}}(q^{-2\lambda_2};q^2)_l(q^{-2(\lambda_1+\lambda_2+1)};q^2)_l.$$

Note that this expression is positive for $0 \le l \le \lambda_2$ and equals zero for $l > \lambda_2$. Collecting all the intermediate results gives the explicit expression for the norm of the basis elements. □

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